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Exploring the Role of Hunting Cooperation, and Fear in a Prey-Predator Model with Two Age Stages

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Abstract

The aim of this study is to utilize the behavior of a mathematical model consisting of three-species with Lotka Volterra functional response with incorporating of fear and hunting cooperation factors with both juvenile and adult predators. The existence of equilibrium points of the system was discussed the conditions with variables. The behavior of model referred by local stability in nearness of any an equilibrium point and the conditions for the method of approximating the solution has been studied locally. We define a suitable Lyapunov function that covers every element of the nonlinear system and illustrate that it works. The effect of the death factor was observed in some periods, leading to non-stability. To confirm the theoretical findings, practical validation was conducted using a numerical simulation implemented in Mathematica software to prove the validity of what has been proven.

Keywords: hunting cooperation; functional response; fear; Lyapunov function.

1 Introduction

Mathematical method to study the dynamics of relationships between prey and predator with existence predators and their young is called age stage-structure prey predator model which provides an effective basis for capturing the complicated nature of prey-predator dynamic in environment since it takes into account the life history stages of both prey and predator species, it provides a more extensive examination of population dynamics in prey-predator models. In addition to requirements continuous models, stage structure models offer a better representation of population dynamics by separating the population apart into discrete stages, for example adult and juvenile, and taking into account changes cross these stages.

In general, these models consists of system of differential equations involving variables such birth rates, growth rates, and death rates to represent the dynamics of every population stage, moreover, the study of complex ecological phenomena, which include the effects of predators on prey action, through our study, we observed in Promrak et al. [20] by using the stage structure, the model provides two stable states, one that includes a single species and the other containing of several coexistence species. Droz and Pękalski [9] explored the effect of predator mobility and hunting capability on the system dynamics. Anti-predator was also studied by [19, 10] to reduced their death and increase their possibility of survival. Ibrahim and Naji [13] discussed stage structure in food web with three species by using Lotka-Voltera functional response.

Some research has added external food sources as well as a shelter factor [12]. The previous study offered understanding of the importance of cooperative hunting. We additionally saw an existence of research on fear and the effect it has on the model, [2] focused on examining the dynamic interactions within the ecosystem comprising prey, predators, and scavengers, given the critical role of scavengers in environmental sustainability and waste management. [7, 11] discussed growing the prey's fear rate results in a decline of the predator population, but the system remains at the coexistence equilibrium point, the location of which varies according to the prey's fear rate value. Additional investigators examined mathematical modeling incorporates biological aspects such as fear of predation, refugees, and harvesting to analyze their impact on the dynamic behavior of the proposed system [17, 28]. Studying the effect of fear on prey predators and determining what causes the effect also investigated in many as in [26, 15].

Consuming on species may directly lower their population number. The decrease in the quantity of prey might result in a reduction of supplies of food for predators, which could lead to a drop in predator populations if prey is the only source of food. The analysis of harvesting in prey predator model covers several aspects, including the scale and duration of harvesting [6, 24].

In addition, [21] researched the relation among predator hunting with prey antipredator action in environment, it tested a stochastic predator-prey model which included fear consequences and hunting cooperation. Another researcher [8, 22] pointed out the technique for diffusion along a prey-predator connection which refers to geographical relocation that occurs among population and the impact it has on population connections. Both [4, 27] discussed diffusion with allee effect and hunting cooperation without delayed and with delayed respectively. [16] studied the dynamical behavior of refuge and hunting cooperation on intraguild prey-predator model. One group of researchers examined the Alee effecting, as referred in [25, 18], while a different group of researchers analyzed the stability by using fuzzy impulsive control, as discussed in [23].

Other papers have been reviewed that attempt to address the problem of disease presence in the model [1, 14]. The objective is to examine the interactions between diseases and prey-predator models and analyses what this means for ecological and epidemiological research [5, 3].

The novelty of the research can be summarized as follows: The presence of the factor of fear in the prey from the predator, because of the impact of cooperative hunting on wildlife in a large way, leads to the extinction of some animal species, and this is what was observed in the numerical drawings. It negatively affects the food chain and the balance of the ecosystem in general. Overhunting leads to the extinction of some animal species, affecting biodiversity and reducing livestock wealth.By examining the model proposed in this study, we gained insight into the effect of cooperative hunting on prey and how to identify the parameters that affect the system and conservation of wildlife. This includes determining hunting seasons and permitted quantities and maintaining the ecological balance.

2 Hunting Cooperation in Prey-Predator Model

This section examined the mathematical model of the Lotka-Volterra prey-predator system, made up of one prey species $\mathcal{X}(\mathcal{T})$, and Juvenile and Adult predator $\mathcal{Y}(\mathcal{T})$, $Z(\mathcal{T})$ at time $\mathcal{T}$ respectively. The three-dimensional Lotka-Volterra model represents the logistical expansion of three species along with the carrying capacity element. One could use the following three first-order nonlinear differential equations to describe the dynamics of the food web system:

$$\begin{aligned}\frac{d\mathcal{X}}{d\mathcal{T}} &= \frac{r\mathcal{X}}{1+s\mathcal{Z}}\left[1-\frac{\mathcal{X}}{\mathcal{K}}\right]-(\beta+\alpha\mathcal{Z})\,\mathcal{X}\mathcal{Z},\\ \frac{d\mathcal{Y}}{d\mathcal{T}} &= e\,(\beta+\alpha\mathcal{Z})\,\mathcal{X}\mathcal{Z}-a\mathcal{Y}-d_1\mathcal{Y},\\ \frac{d\mathcal{Z}}{d\mathcal{T}} &= a\mathcal{Y}-d_2\mathcal{Z}.\end{aligned} \tag{1}$$

With $\mathcal{X}(0)$, $\mathcal{Y}(0)$, and $\mathcal{Z}(0)$ are all positive. In absence of predation, the growth rate of prey increases logistically with carrying capacity $\mathcal{K}$. The description of all parameters which control prey and population react of (1) are:

- r: The rate at which the prey population grows naturally when there is no risk from predators is known as the intrinsic growth rate of prey.
- s: The fear rate of prey is an estimate of how quickly prey notices and responds to the threat of predators $\mathcal{K} > 0$.
- $\mathcal{K}$: The maximum population size that the ecosystem can support is determined by the carrying capacity of prey.
- β: The attack rate, which defines the rate that predators catch and consume each those killed.
- α: The degree of collaboration, which affects how well predators cooperate in order to catch and hunt prey.
- e: The rate that adult prey produces juvenile predators, or the efficiency in which the biomass of prey undergo a transformed the biomass of juvenile predators.
- a: Rate that adult predator become juvenile predators, or the accuracy in which the biomass of prey undergoes a transformed the biomass of juvenile predators.
- d_1: Juvenile predator death rate represents the rate at which these predators die out because of numerous mortality causes.

d_2: The adult predator death rate is the number of adult predators which expire as the consequence of disease, circumstances, and predation.

In order to simplify the analysis process, we continue to non-dimensionless (1) by eliminating all units from variables by benefit from applying of the next hypotheses;

$$\begin{aligned} &t = r\mathcal{T}, \quad && x = \frac{\mathcal{X}}{\mathcal{K}}, \quad && y = \frac{\beta}{r}\mathcal{Y}, \quad && z = \frac{\beta}{r}\mathcal{Z}, \\ &w_0 = s\frac{r}{\beta}, \quad && w_1 = \frac{\alpha}{\beta^2}r, \quad && w_2 = e\frac{k}{r}\beta, \\ &w_3 = \frac{a}{r}, \quad && w_4 = \frac{d_1}{r}, \quad && w_5 = \frac{d_2}{r}. \end{aligned} \tag{2}$$

Thus, the non-dimensional system associated with (1) can be expressed in the given format;

$$\begin{aligned} \frac{dx}{dt} &= \left(\frac{(1-x)}{1+w_0 z} - z\left(1+w_1 z\right)\right)x = x f_1\left(x,y,z\right), \\ \frac{dy}{dt} &= w_2\left(1+w_1 z\right)xz - w_3 y - w_4 y = f_2(x,y,z), \\ \frac{dz}{dt} &= w_3 y - w_5 z = f_3\left(x,y,z\right). \end{aligned} \tag{3}$$

The functions that given by the vector $f = (f_1, f_2, f_3)^T$ for $(x, y, z) \in R^3$ with positive values for all $x\left(t\right), y\left(t\right), z\left(t\right)$ in the right-hand side of the (3) are continuous partial derivative on R^3_+. In addition, the solution of (3) exists and unique. So, they are Lipschitz functions.

Theorem 2.1. *Every one of the solutions of system (3) are uniformly bounded which beginning in R^3_+.*

Proof. Assume that $y\left(t\right), x\left(t\right)$, and $z\left(t\right)$ any solution of (3), from prey equation we get

$$\frac{dx}{dt} \leq \frac{x}{1+w_0 z}\left(1-x\right) \leq x\left(1-x\right).$$

Subsequently by resolving the previously discussed inequality, It has been established $x \leq 1$; $t \to \infty$. Define the function $L\left(t\right) = x\left(t\right) + \frac{1}{w_2}y(t) + \frac{1}{w_2}z(t)$, the we get the following result;

$$\frac{dL}{dt} \leq 2x - x - \frac{1}{w_2}\Big[w_4 y + w_5 z\Big].$$

Then,

$$\frac{dL}{dt} \leq 2 - \delta\left[x + \frac{1}{w_2}y + \frac{1}{w_2}z\right] \leq 2 - \delta L,$$

where

$$\delta = \min\{1, w_4, w_5\}.$$

By using the Gronwall inequality gives $\frac{dL}{dt} + \delta L \leq 2$, then we get $L \leq \frac{2}{\delta}$ as $t \to \infty$. Hence, all solutions of (3) that initiating in R^3_+ are uniformly bounded. □

3 Local Stability Analysis

In this section, we can see that (3) has most three non-negative equilibrium points, the existence conditions of these equilibrium points and stability analyses are carried out. The points have been marked as follows;

- The points $P_0 = (0, 0, 0)$, and $P_x = (1, 0, 0)$, which is constantly present.
- The positive equilibrium point $P_{xyz} = (\tilde{x}, \tilde{y}, \tilde{z})$, where

$$\left.\begin{aligned} \tilde{x} &= 1 - \tilde{z}(w_0\tilde{z} + 1)(w_1\tilde{z} + 1), \\ \tilde{y} &= \frac{w_5}{w_3}\tilde{z}. \end{aligned}\right\} \tag{4}$$

As for $\tilde{z}$, it was obtained from five-order polynomial equation,

$$H_1 z^5 + H_2 z^4 + H_3 z^3 + H_4 z^2 + H_5 z = 0,$$

where

$$\begin{aligned} H_1 &= -w_0 w_1^2 w_2, \\ H_2 &= -2w_0 w_1 w_2 - w_1^2 w_2, \\ H_3 &= -2w_1 w_2 - w_0 w_2, \\ H_4 &= w_1 w_2 - w_2, \\ H_5 &= w_2 - w_5 - \frac{w_4 w_5}{w_3}. \end{aligned}$$

Clearly, the positive equilibrium point $P_{xyz} = (\tilde{x}, \tilde{y}, \tilde{z})$ exists uniquely in the interior of R_+^3 under the following condition,

$$w_2 > w_5 + \frac{w_4 w_5}{w_3}, \qquad w_1 < 1, \tag{5}$$

presently, let's evaluate the stability analysis of all equilibrium points given previously, we used the Jacobian matrix, which has been stated by $J(x, y, z)$ and then determined the eigenvalues of them. It is easy to verify that the Jacobian matrix of (3) at the point (x, y, z) can be represented as,

$$J = \begin{bmatrix} xf_{1x} + f_1 & xf_{1y} & xf_{1z} \\ f_{2x} & f_{2y} & f_{2z} \\ f_{3x} & f_{3y} & f_{3z} \end{bmatrix}, \tag{6}$$

where

$$\begin{aligned} f_{1x} &= -\frac{1}{1 + w_0 z}, & f_{1y} &= 0, & f_{1z} &= -\frac{w_0}{(1 + w_0 z)^2}(1 - x) - (1 + 2w_1 z), \\ f_{2x} &= w_2(1 + w_1 z)z, & f_{2y} &= -w_3 - w_4, & f_{2z} &= w_2(1 + 2w_1 z)x, \\ f_{3x} &= 0, & f_{3y} &= w_3, & f_{3z} &= -w_5. \end{aligned} \tag{7}$$

The Jacobian matrix for the point $P_0 = (0, 0, 0)$, can be earned as follow,

$$J(P_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -w_3 - w_4 & 0 \\ 0 & w_3 & -w_5 \end{bmatrix}. \tag{8}$$

Then, there are three eigenvalues of $J(P_0)$, $\lambda_{11} = 1 > 0$, $\lambda_{12} = -w_3 - w_4 < 0$, and $\lambda_{13} = -w_5 < 0$. Hence, there is a positive equilibrium point and the other two equilibrium points are negative, so that P_0 is saddle point.

Consequently, for the one-species equilibrium point $P_x = (1, 0, 0)$, the Jacobian matrix can be obtained as the following,

$$J(P_x) = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -w_3 - w_4 & w_2 \\ 0 & w_3 & -w_5 \end{bmatrix}. \tag{9}$$

As a result, we can write the characteristic equation of $J(P_x)$ as follows,

$$\left[-\lambda_{21} - 1\right]\left[\lambda^2 - T_x\lambda + D_x\right] = 0, \tag{10}$$

where

$$T_x = -(w_3 + w_4 + w_5) < 0.$$
$$D_x = (w_3 + w_4)\, w_5 - w_2 w_3.$$

Hence, there are three eigenvalues of

$$J(P_x), \qquad \lambda_{21} = -1 < 0, \qquad \lambda_{22} = \frac{T_x}{2} - \frac{1}{2}\sqrt{{T_x}^2 - 4D_x},$$

and

$$\lambda_{23} = \frac{T_x}{2} + \frac{1}{2}\sqrt{{T_x}^2 - 4D_x}.$$

Clearly, if and only if the following conditions hold, then P_x is a locally asymptotical stable point,

$$\big(w_3 + w_4\big)w_5 > w_2 w_3. \tag{11}$$

Theorem 3.1. *Suppose that the positive equilibrium point $P_{xyz} = (\tilde{x}, \tilde{y}, \tilde{z})$ of the system (3) exists, then P_{xyz} is locally asymptotically stable if and only if the following conditions hold;*

$$\left[l_{22}l_{33} - l_{23}l_{32}\right] > 0, \tag{12}$$
$$2l_{11}l_{22}l_{33} < l_{21}l_{13}l_{32}. \tag{13}$$

Proof. For the positive equilibrium point $P_{xyz} = (\tilde{x}, \tilde{y}, \tilde{z})$, the Jacobian matrix can be obtained as the following,

$$J(P_{xyz}) = [l_{ij}]_{3\times 3},$$

where

$$l_{11} = -\frac{\tilde{x}}{1 + w_0\tilde{z}}, \qquad l_{12} = 0, \qquad l_{13} = -\tilde{x}\left(\frac{w_0}{(1 + w_0\tilde{z})^2}(1 - \tilde{x}) + (1 + 2w_1\tilde{z})\right),$$
$$l_{21} = w_2(1 + w_1\tilde{z})\,\tilde{z}, \qquad l_{22} = -(w_3 + w_4), \qquad l_{23} = w_2(1 + 2w_1\tilde{z})\,\tilde{x},$$
$$l_{31} = 0, \qquad l_{32} = w_3, \qquad l_{33} = -w_5.$$

In this case, the characteristic equation is as follows,

$$\lambda^3 + L_1\lambda^2 + L_2\lambda + L_3 = 0, \tag{14}$$

where

$$L_1 = -\left[l_{11} + l_{22} + l_{33}\right],$$
$$L_2 = l_{11}l_{22} + l_{11}l_{33} + l_{22}l_{33} - l_{23}l_{32},$$
$$L_3 = -\left[l_{11}l_{22}l_{33} + l_{21}l_{13}l_{32} - l_{11}l_{23}l_{32}\right].$$

While,

$$\begin{aligned}\Delta &= L_1L_2 - L_3,\\ &= -(l_{11}l_{22})(l_{11} + l_{22}) - (l_{11}l_{33})(l_{11} + l_{33}) - (l_{22}l_{33})(l_{22} + l_{33})\\ &\quad - 2l_{11}l_{22}l_{33} + (l_{23}l_{32})(l_{22} + l_{33}) + l_{21}l_{13}l_{32}.\end{aligned}$$

The Routh-Hurtwitz criterion states that , all roots of the characteristic (14) contain negative real parts, then under the satisfied conditions (12), (13), the positive equilibrium point, $P_{xyz} = (\tilde{x}, \tilde{y}, \tilde{z})$ is locally asymptotically stable if and only if $L_1 > 0$, $L_3 > 0$ and $\Delta > 0$. □

4 The Analysis of Global Stability

This section describes each equilibrium point of a (3) the basin of attraction that belong to the $int\ R_3^+$ is investigated by choosing the suitable the Lyapunov function, as shown in the following theorems.

Theorem 4.1. *In the presence of sufficient condition,*

$$\frac{w_5}{w_2} > (1 + w_1 z_{\max}) > 1 + w_1 z, \tag{15}$$

in $int\ R_3^+$, if the point P_x is locally asymptotically stable, then P_x is also globally asymptotically stable within $int\ R_3^+$.

Proof. Let us define the following Lyapunove function, $\eta_1 = \left(x - \bar{x} - \bar{x}\ \ln\frac{x}{\bar{x}}\right) + \frac{y}{w_2} + \frac{z}{w_2}$, with $\bar{x} = 1$. Obviously, $\eta_1 : R_3^+ \to R$ is a continuously differentiable positive define real-valued function that is $\eta_1(P_x) = 0$, while $\eta_1(x, y, z)$ positive defined for $\forall\ (x, y, z) \neq P_x, (x, y, z) \in R_3^+$, $y \geq 0$, $x > 0$, $z \geq 0$, then differentiating η_1 with respect to the time t, then we get,

$$\begin{aligned}\frac{d\eta_1}{dt} &= \left(\frac{x - \bar{x}}{x}\right)\frac{dx}{dt} + \frac{1}{w_2}\frac{dy}{dt} + \frac{1}{w_2}\frac{dz}{dt},\\ &= -\frac{(x-1)^2}{1 + w_0 z} + (1 + w_1 z)\,z - \frac{w_4}{w_2}y - \frac{w_5}{w_2}z\\ &\leq -\frac{(x-1)^2}{1 + w_0 z} - \frac{w_4}{w_2}y - \left[\frac{w_5}{w_2} - (1 + w_1 z)\right]z.\end{aligned}$$

Therefore, the sufficient condition (15) guarantee that $\frac{d\eta_1}{dt}$ is less than zero. Hence, the point P_x is a globally asymptotically stable. □

Theorem 4.2. *In the presence of sufficient conditions,*

$$(\gamma_4)^2 < \gamma_1\gamma_2, \tag{16}$$
$$(\gamma_5)^2 < \gamma_1\gamma_3, \tag{17}$$
$$(\gamma_6)^2 < \gamma_2\gamma_3, \tag{18}$$

in $int\ R_3^+$, $if\, P_{xyz}$ is locally asymptotically stable, then P_{xyz} is also globally asymptotically stable.

Proof. Let us define the following Lyapunove function,

$$\eta_2 = \left(-\bar{\bar{x}} + x - \bar{\bar{x}} \ln \frac{x}{\bar{\bar{x}}}\right) + \frac{(y-\bar{\bar{y}})^2}{2} + \frac{(z-\bar{\bar{z}})^2}{2}.$$

Obviously, $\eta_2 : R_3^+ \to R$ is a continuously differentiable positive define real-valued function that is $\eta_2\left(P_{xyz}\right) = 0$, while $\eta_2(x,y,z)$ is positive defined for $\forall\ (x,y,z) \neq P_{xyz}$, $(x,y,z) \in R_3^+$.

Given $y \geq 0$, $x > 0$, $z \geq 0$, then differentiating η_2 with respect to the time t, then we get,

$$\begin{aligned}
\frac{d\eta_2}{dt} &= \left(\frac{x-\bar{\bar{x}}}{x}\right)\frac{dx}{dt} + (y-\bar{\bar{y}})\frac{dy}{dt} + (z-\bar{\bar{z}})\frac{dz}{dt},\\
&= -\frac{(x-\bar{\bar{x}})^2}{1+w_0 z} + \left[w_2 z\left(1+w_1 z\right)\right](y-\bar{\bar{y}})(x-\bar{\bar{x}}) - (w_3+w_4)(y-\bar{\bar{y}})^2\\
&\quad - \left[\frac{w_0\left(1-\bar{\bar{x}}\right)}{(1+w_0 z)\left(1+w_0\bar{\bar{z}}\right)} + \left(1+w_1\left(z+\bar{\bar{z}}\right)\right)\right](z-\bar{\bar{z}})(x-\bar{\bar{x}}) - w_5(z-\bar{\bar{z}})^2\\
&\quad + \left[w_2\bar{\bar{x}}\left(w_1\left(z+\bar{\bar{z}}\right)+1\right)+w_3\right](z-\bar{\bar{z}})(y-\bar{\bar{y}}).
\end{aligned}$$

Then, by using the giving conditions, we have that,

$$\begin{aligned}
\frac{d\eta_2}{dt} &= -\frac{1}{2}\left[\sqrt{\gamma_1}\,(x-\bar{\bar{x}}) - \sqrt{\gamma_2}\,(y-\bar{\bar{y}})\right]^2 - \frac{1}{2}\left[\sqrt{\gamma_1}\,(x-\bar{\bar{x}}) + \sqrt{\gamma_3}\,(z-\bar{\bar{z}})\right]^2\\
&\quad - \frac{1}{2}\left[\sqrt{\gamma_2}\,(y-\bar{\bar{y}}) - \sqrt{\gamma_3}\,(z-\bar{\bar{z}})\right]^2.
\end{aligned}$$

Therefore, the sufficient conditions (16), (17) and (18) guarantees that, $\frac{d\eta_2}{dt}$ is less than zero. Hence, the positive equilibrium point, $P_{xyz} = (\tilde{x}, \tilde{y}, \tilde{z})$ is a globally asymptotically stable. □

5 Numerical Analysis

The numerical evaluation is a mathematical method performed in order to understand and evaluate the dynamics of prey-predator systems. It includes the application of mathematical models and algorithms to figure out and evaluate problems found in these systems. The key goal of scientific research in predator-prey systems is to gain an understanding the dynamics of the system, including its functioning and evolution, as well as the potential impacts of human activities on it. Numerical analysis is an efficient instrument in this field as it allows scientists to simulate and analyze large and complex data sets, and to make accurate assumptions about the behavior of the system. Therefore, we will study our system and apply the data that was estimated and selected as appropriate as possible to obtain the approach points and the selected data are shown in the Table 1.

Table 1: Data of parameter values.

w_0	w_1	w_2	w_3	w_4	w_5
0.7	0.7	0.6	0.3	0.3	0.2

Table 1 contains six parameters, which were replaced in (3), we obtained a global asymptotical stable point $(0.549, 0.204, 0.306)$. In Figure 1, drawing for the prey, juvenile and adult predator from initial point $(0.5, 0.9, 0.7)$ in (a) three-dimension phase portrait and time series in (b).

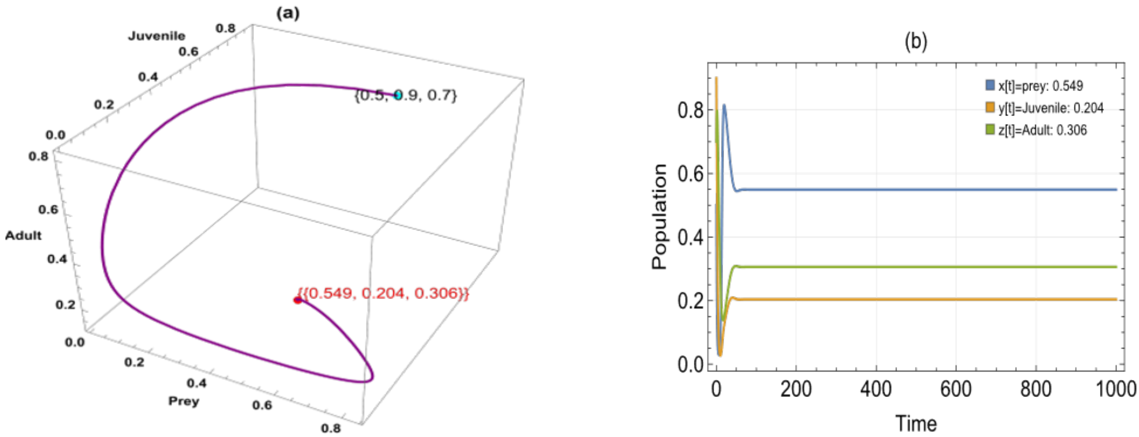
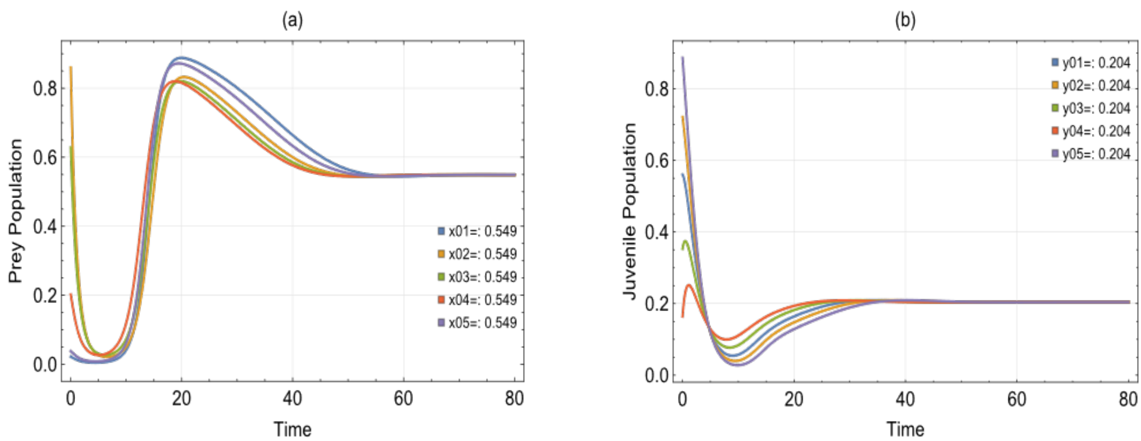


Figure 1: (**a**) $3\mathcal{D}$ -Phase portrait of the (3) (**b**) The time series of the (3) by utilizing Table 1, the trajectories of three species demonstrate an asymptotic positive convergence towards $P_{xyz} = (0.549, 0.204, 0.306)$.

In Figure 2, we chose five initial points. It was found that the approximate solution reached stability three times: once for the prey (a), once for the juvenile (b), and once for the adult (c). In (d), the whole population is collected together into a time series, and in (e), it is represented in three-dimension phase portrait.



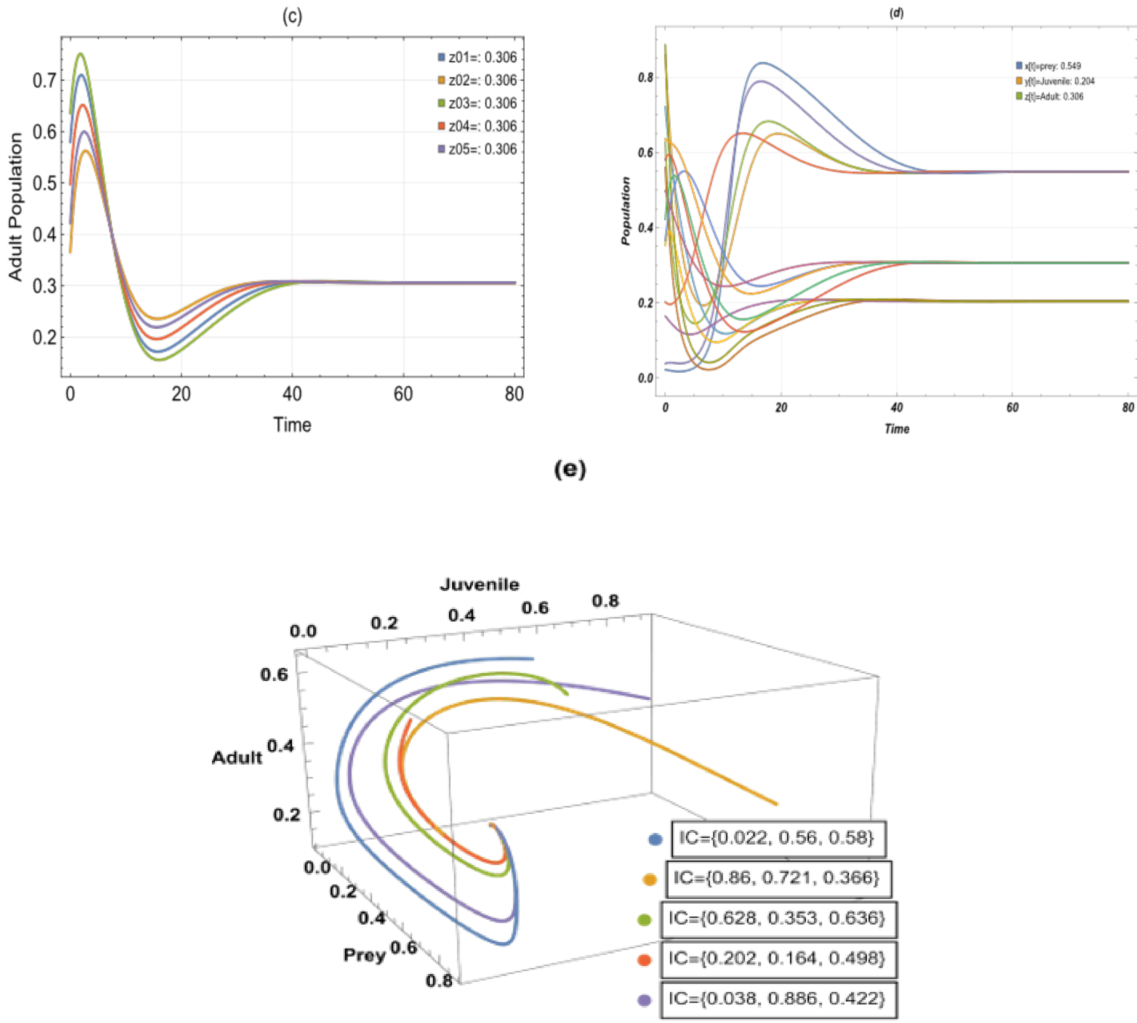


Figure 2: (**a**) The prey population starting from different initial points. (**b**) The juvenile predator population starting from different initial points. (**c**) The adult predator starting from different initial points. (**d**) The time series exhibits the trajectories of the 3, for population from five different initial start. (**e**) $3\mathcal{D}$-Phase portrait of the (3), for five distinct beginning , the convergence is towards $P_{xyz} = (0.549, 0.204, 0.306)$.

The parameters w_0, w_1, have a quantitative impact on the (3), consequently, we will only examine the parameters that influence the system. The parameter w_2 approaches a positive point within the interval $(0.387, 1)$, while within interval $(0, 0.387]$ it approaches to P_x, as represented in Figure 3.

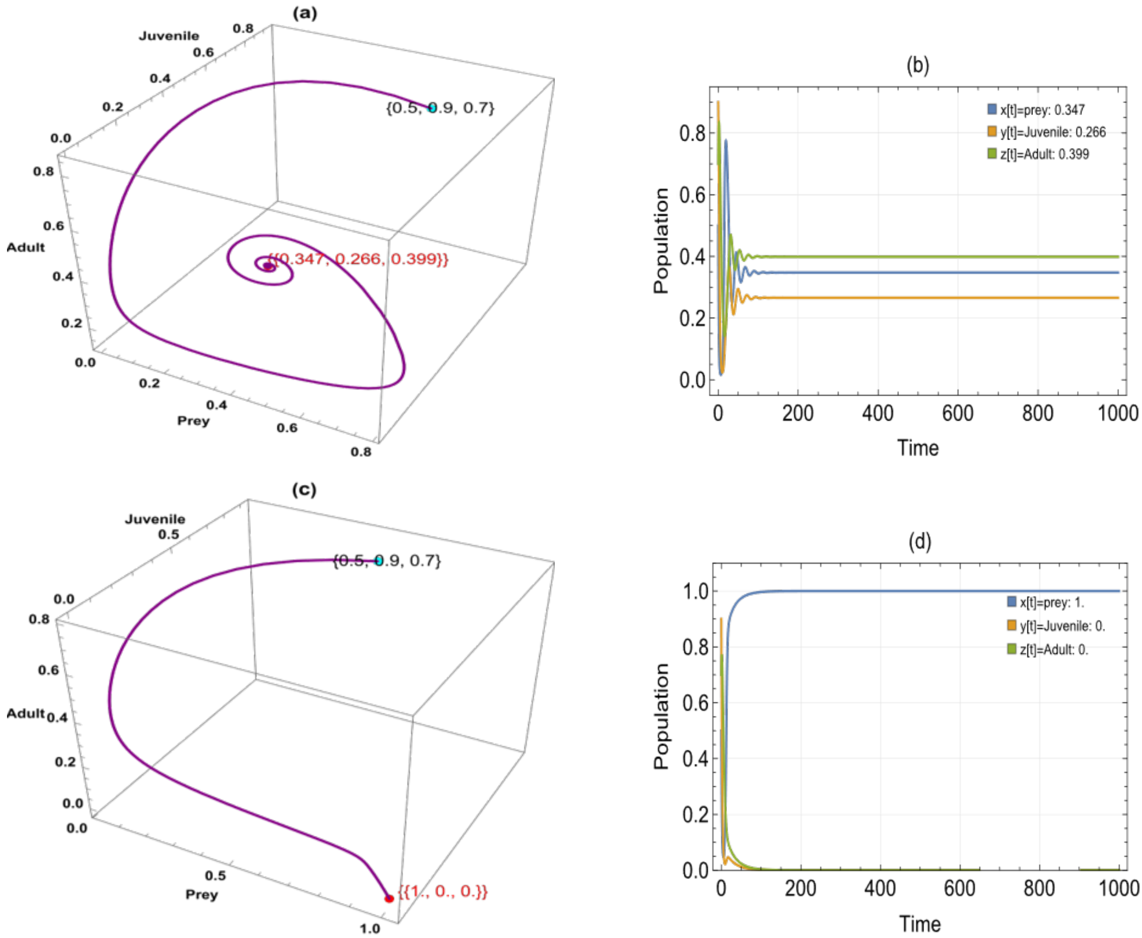


Figure 3: (**a**) Trajectories of system approach asymptotically to P_{xyz}. (**b**) Time series of the (3) converge asymptotically to $P_{xyz} = (0.347,\ 0.266\ , 0.399)$ for $w_2 = 0.9$. (**c**) Trajectories of system converge asymptotically to P_x. (**d**) Time series of the (3), converge asymptotically to $P_x = (1, 0, 0)$ for $w_2 = 0.3$.

When plotting the outcome of changing the w_3, three cases appear during certain periods. The first case is approaching the point P_{xyz} during the period $0.141 < w_3 < 1$, the second case is approaching P_x during the period $0 < w_3 \leq 0.141$, as represented in Figure 4.

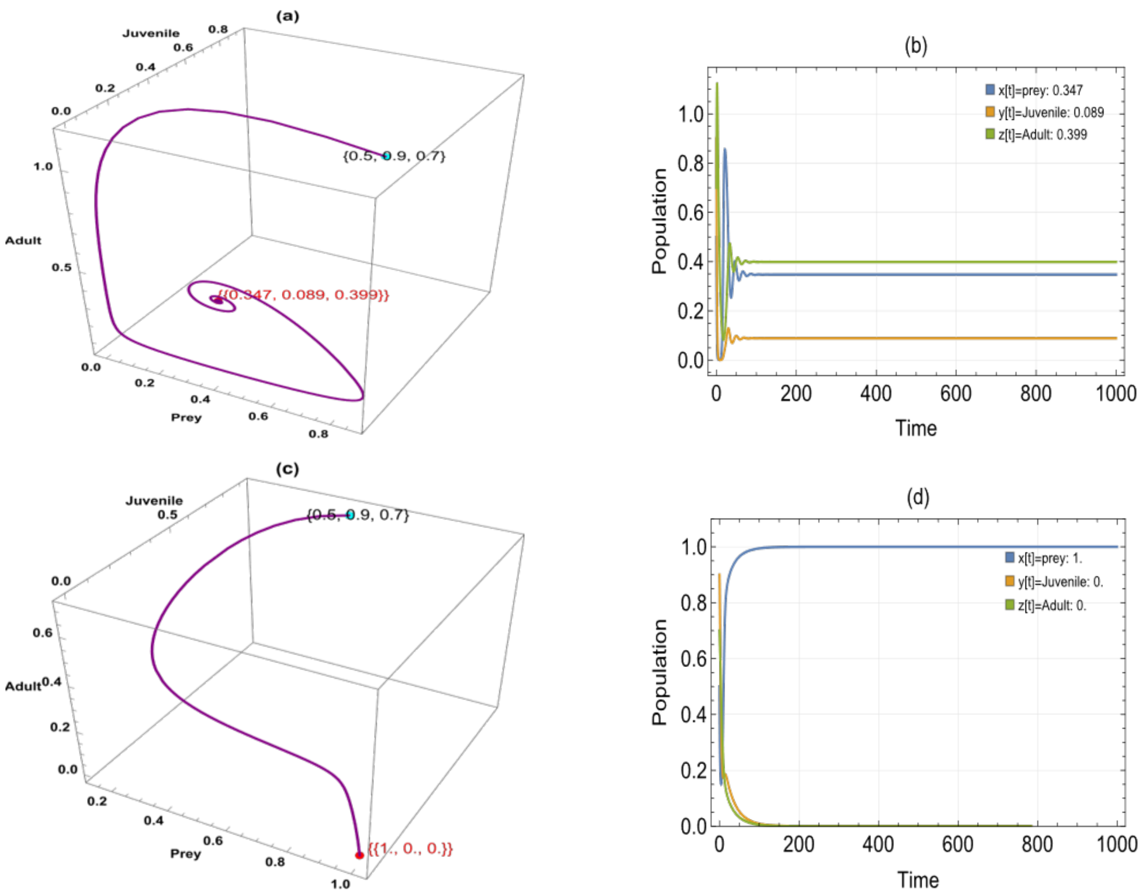


Figure 4: (**a**) Trajectories of system approach asymptotically to P_{xyz}. (**b**) Time series of the (3) converge asymptotically to $P_{xyz} = (0.347, 0.089, 0.399)$ for $w_3 = 0.9$. (**c**) Trajectories of system converge asymptotically to P_x. (**d**) Time series of the (3), converge asymptotically to $P_x = (1, 0, 0)$ for $w_3 = 0.1$.

When plotting the outcome of changing the w_4, three cases appear during certain periods. The first case is approaching the point P_{xyz} during the period $0.01 < w_4 < 0.626$, the second case is approaching P_x during the period $0.626 \leq w_4 < 1$, as represented in Figure 5.

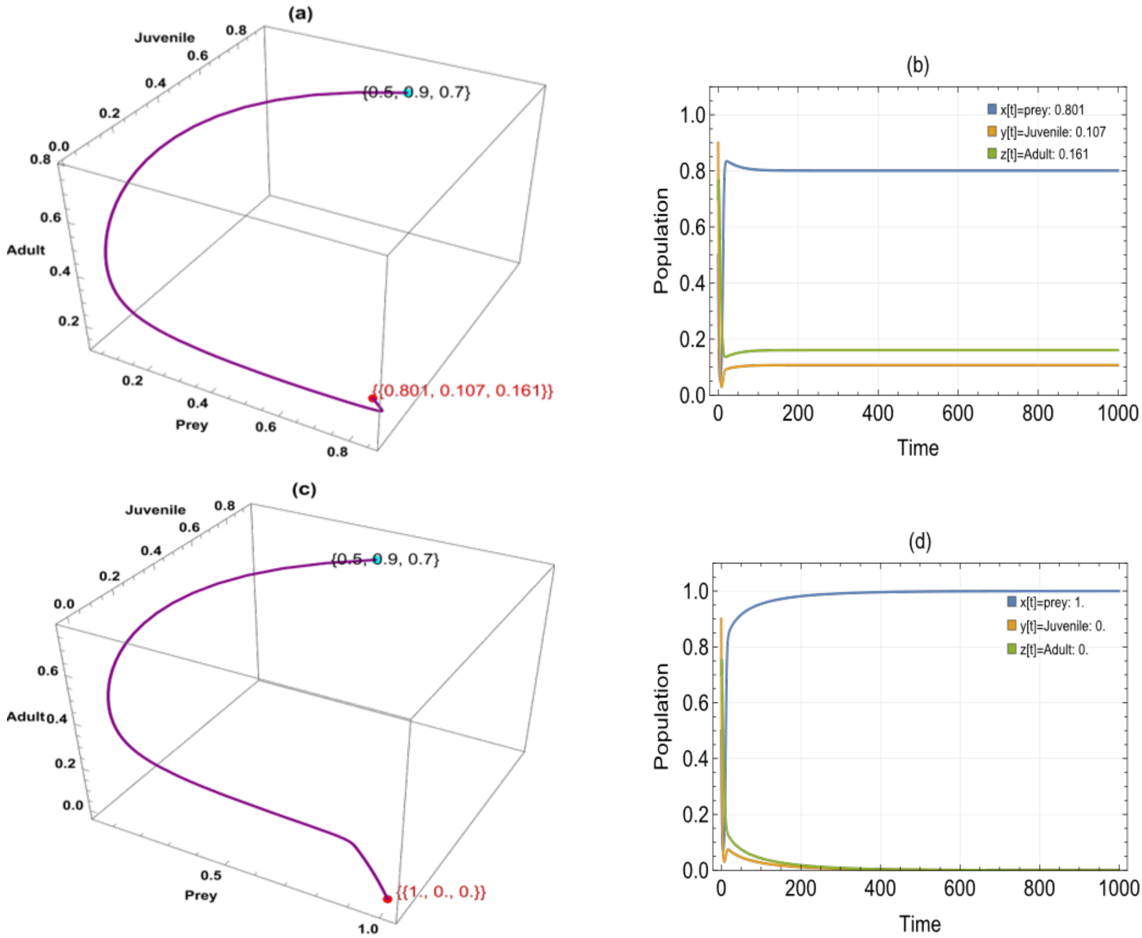


Figure 5: (**a**) Trajectories of system approach asymptotically to P_{xyz}. (**b**) Time series of the (3) converge asymptotically to $P_{xyz} = (0.801, 0.107, 0.161)$ for $w_4 = 0.502$. (**c**) Trajectories of system converge asymptotically to P_x. (**d**) Time series of the (3), converge asymptotically to $P_x = (1, 0, 0)$ for $w_4 = 0.64$.

When plotting the outcome of changing the w_5, three cases appear during certain periods. The first case is approaching the point P_{xyz} during the period $0.03 \leq w_5 < 0.307$, the second case is approaching P_x during the period $0.307 \leq w_5 < 1$, as represented in Figure 6.

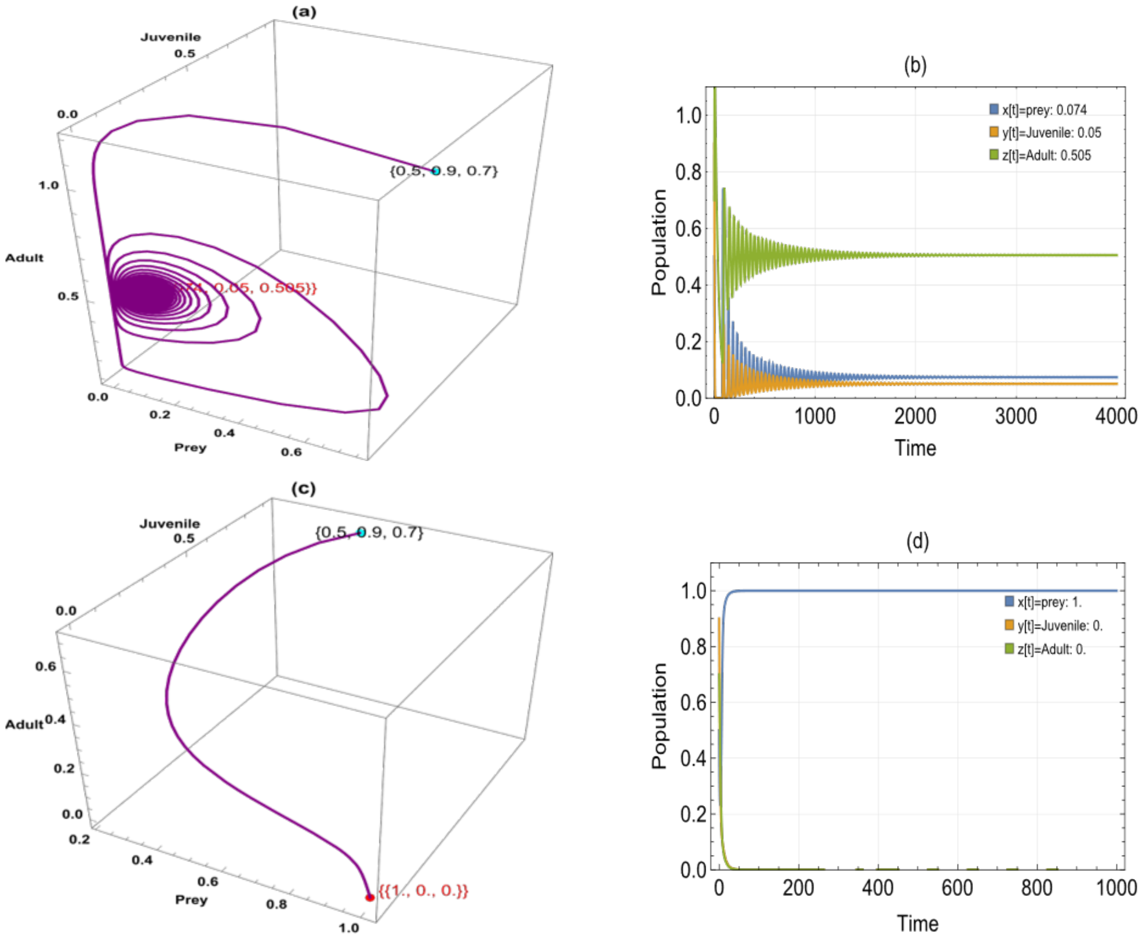


Figure 6: (**a**) Trajectories of system approach asymptotically to P_{xyz}. (**b**)Time series of the (3) converge asymptotically to $P_{xyz} = (0.074, 0.05, 0.505)$ for $w_5 = 0.03$. (**c**) Trajectories of system converge asymptotically to P_x. (**d**) Time series of the (3), converge asymptotically to $P_x = (1, 0, 0)$ for $w_5 = 0.5$.

6 Conclusions

This study examined the roles of fear in relationships with prey and predators. The sense of fear can have an important effect on the action of creatures of prey, changing their habits of seeking out food, their motions, and the selection of territory. Further, fear may affect the behavior of predators, causing alterations in hunting methods and prey choice. So, we investigated the purpose of hunting cooperation in predator-prey relationships and the consequences of fear into mathematical models. These models offer helpful understandings into the dynamics of prey-predator systems and have improved our awareness of the effects of fear and hunting cooperation on population dynamics, ecological stability, and all variety.

Simulation was used to study a mathematical model to understand the behavior of the dynamic system. Through drawings, it was possible to approximate the behavior of dynamic systems and predict their future behavior.

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Conflicts of Interest The authors declare no conflict of interest.

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